APPROXIMATE THEORY OF BUCKLING OF THIN PLATES OF SEMILINEAR MATERIAL IN CASE OF AFFINE INITIAL DEFORMATION

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L. M. ZUBOV
(Leningrad)

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Approximate two-dimensional equations are constructed which describe the buckling of thin nonlinearly elastic plates in case of affine initial deformation. A description of semilinear material is adopted [1] which is a generalization of Hooke's law to the case of finite deformations. At first the variational formulation of the problem of bifurcation of equilibrium is given for semilinear material. For the case of affine initial deformation the mixed variational principle is formulated. From this, two-dimensional equations of neutral equilibrium are derived by means of approximation of variation of unknown functions with thickness. Using the buckling of a circular plate compressed along the contour by uniform pressure as an example, a comparison of results with classical linear theory of plate buckling is carried out. For a circular cylinder results are also compared with the exact solution obtained by Sensenig. Notations of vector and tensor quantities describing the nonlinearly elastic medium are taken from [1].

1. Energy criterion of bifurcation of equilibrium for semilinear material. The potential energy of an elastic body in the absence of volume forces and with "dead" surface loading is written in the form

$$\Pi = \iiint\limits_{v} W\, d\tau - \iint\limits_{o} \mathbf{F}^\circ \cdot \mathbf{u}\, do$$

Here $\mathbf{u}$ is the displacement vector. Integration is performed over the volume v and the surface o of the body in the undeformed state. For the "semilinear" material

$$W = {}^1/_2 \lambda s_1^2 + \mu s_2$$

$$s_1 = I_1 (\mathrm{G}^{\times 1/2}) - 3, \qquad s_2 = I_1 (\mathrm{G}^\times) - 2 I_1 (\mathrm{G}^{\times 1/2}) + 3 \tag{1.1}$$

Here $\mathrm{G}^\times$ is Cauchy's measure of deformation. $I_1 (\mathrm{G}^\times)$ is its first invariant, λ and μ are constants.

Let us examine an initial deformed state of the body with a radius vector of the point of the elastic body $\mathbf{R}^\circ$, and adjacent to this the point given by the vector

$$\mathbf{R} = \mathbf{R}^\circ + \eta \mathbf{w}$$

where η is the small parameter.

In order to obtain the variational formulation of the problem on bifurcation of the equilibrium, it is necessary [2] to compute the growth increment of potential energy when the additional displacement $\eta\mathbf{w}$ is communicated to the points with accuracy to terms of second order of smallness

$$\Pi = \Pi_0 + \eta \Pi_1 + \eta^2 \Pi_2 + \ldots \qquad (\Pi_0 = \Pi (\mathbf{R}^\circ))$$

From (1.1) we obtain
$$W - W_0 = (\lambda s_1^\circ - 2\mu) I_1 (\mathrm{G}^{\times 1/2} - \mathrm{G}^{\times \circ 1/2}) + $$
$$+ {}^1/_2 \lambda I_1^2 (\mathrm{G}^{\times 1/2} - \mathrm{G}^{\times \circ 1/2}) + \mu I_1 (\mathrm{G}^\times - \mathrm{G}^{\times\circ}) \tag{1.2}$$

Since $G^\times = \nabla\mathbf{R}\cdot\nabla\mathbf{R}^T$, then

$$G^\times - G^{\times\circ} = \eta(\nabla\mathbf{R}^\circ\cdot\nabla\mathbf{w}^T + \nabla\mathbf{w}\cdot\nabla\mathbf{R}^{\circ T}) + \eta^2\nabla\mathbf{w}\cdot\nabla\mathbf{w}^T$$

Referring to equation [1]

$$\nabla\mathbf{R} = \sqrt{G_k}\,\mathbf{e}_k\mathbf{e}_k'$$

where G_k are the principal values of tensor $G^\times$; $\mathbf{e}_k$ and $\mathbf{e}_k'$ are basis vectors of principal directions of Cauchy's measure of deformation and of Almansi's measure of deformation, respectively, we have

$$I_1(G^\times - G^{\times\circ}) = 2\eta\sqrt{G_k^\circ}\,\mathbf{e}_k^\circ\cdot\nabla\mathbf{w}\cdot\mathbf{e}_k^{\circ\prime} + \eta^2\nabla\mathbf{w}\cdot\cdot\nabla\mathbf{w}^T \tag{1.3}$$

For computation of the quantity $I_1(G^{\times 1/2} - G^{\times\circ 1/2})$ we represent the tensor $G^{\times 1/2}$ in the form

$$G^{\times 1/2} = G^{\times\circ 1/2} + \eta\left(\frac{d}{d\eta}G^{\times 1/2}\right)_{\eta=0} + \frac{1}{2}\eta^2\left(\frac{d^2}{d\eta^2}G^{\times 1/2}\right)_{\eta=0} + \ldots$$

Further we write the identity $\nabla\mathbf{R}\cdot A^T = G^{\times 1/2}$, in which $A = \mathbf{e}_s\mathbf{e}_s'$ is the tensor of rotation of principal axes of deformation, and deferentiate it with respect to parameter η

$$\nabla\mathbf{w}\cdot A^T + \nabla\mathbf{R}\cdot(A^\cdot)^T = (G^{\times 1/2})^\cdot$$

We arrive at the identity

$$I_1^\cdot(G^{\times 1/2}) = \nabla\mathbf{w}\cdot\cdot A^T + \nabla\mathbf{R}\cdot\cdot(\mathbf{A}^\cdot)^T$$

Using the equation [1]

$$(A^\cdot)^T = \frac{\mathbf{e}_k\cdot\nabla\mathbf{w}\cdot\mathbf{e}_s'}{\sqrt{G_k}+\sqrt{G_s}}(\mathbf{e}_s'\mathbf{e}_k - \mathbf{e}_k'\mathbf{e}_s) \tag{1.4}$$

it is easy to prove that $\nabla R\cdot\cdot(A^\cdot)^T \equiv 0$. In fact

$$\nabla\mathbf{R}\cdot\cdot(A^\cdot)^T = \sqrt{G_m}\mathbf{e}_m' \quad (\mathbf{e}_s'\mathbf{e}_k - \mathbf{e}_k'\mathbf{e}_s)\cdot\mathbf{e}_m\frac{\mathbf{e}_k\cdot\nabla\mathbf{w}.\mathbf{e}_s'}{\sqrt{G_s}+\sqrt{G_k}} =$$

$$= \sqrt{G_m}(\delta_{sm}\delta_{km} - \delta_{km}\delta_{sm})\frac{\mathbf{e}_k\cdot\nabla w.\mathbf{e}_s'}{\sqrt{G_s}+\sqrt{G_k}}$$

Here δ_{sm} is the Kronecker symbol. In this manner

$$I_1^\cdot(G^{\times 1/2}) = \nabla\mathbf{w}\cdot\cdot A^T = \mathbf{e}_s\cdot\nabla\mathbf{w}\cdot\mathbf{e}_s'$$

and further according to (1.4)

$$I_1^{\cdot\cdot}(G^{\times 1/2}) = \nabla\mathbf{w}\cdot\cdot(A^\cdot)^T = \frac{\mathbf{e}_k\cdot\nabla\mathbf{w}.\mathbf{e}_s'}{\sqrt{G_s}+\sqrt{G_k}}(\mathbf{e}_k\cdot\nabla\mathbf{w}\cdot\mathbf{e}_s' - \mathbf{e}_s\cdot\nabla\mathbf{w}\cdot\mathbf{e}_k')$$

Thus we obtain with accuracy to small terms of the second order

$$I_1(G^{\times 1/2} - G^{\times\circ 1/2}) = \eta\mathbf{e}_s^\circ\cdot\nabla\mathbf{w}\cdot\mathbf{e}_s^{\circ\prime} +$$

$$+\frac{1}{2}\eta^2\frac{\mathbf{e}_k^\circ\cdot\nabla\mathbf{w}\cdot\mathbf{e}_s^{\circ\prime}}{\sqrt{G_s^\circ}+\sqrt{G_k^\circ}}(\mathbf{e}_k^\circ\cdot\nabla\mathbf{w}\cdot\mathbf{e}_s^{\circ\prime} - \mathbf{e}_s^\circ\cdot\nabla\mathbf{w}\cdot\mathbf{e}_k^{\circ\prime}) \tag{1.5}$$

Substituting (1.3) and (1.5) into (1.2), we arrive at the following relationships:

$$\Pi_1 = \iiint_v\{\nabla\mathbf{w}^T\cdot\cdot[2\mu\nabla\mathbf{R}^\circ + (\lambda s_1^\circ - 2\mu)A^\circ]\}\,d\tau - \iint_{o_1}\mathbf{F}^\circ\cdot\mathbf{w}\,do$$

$$\Pi_2 = \iiint_v\Big[\frac{1}{2}\lambda(\nabla\mathbf{w}\cdot\cdot A^{\circ T})^2 + \mu\nabla\mathbf{w}\cdot\cdot\nabla\mathbf{w}^T + \tag{1.6}$$

$$+\frac{1}{2}\frac{\lambda s_1^\circ - 2\mu}{\sqrt{G_s^\circ}+\sqrt{G_k^\circ}}\mathbf{e}_k^\circ\cdot\nabla\mathbf{w}\cdot\mathbf{e}_s^{\circ\prime}(\mathbf{e}_k^\circ\cdot\nabla\mathbf{w}\cdot\mathbf{e}_s^{\circ\prime} - \mathbf{e}_s^\circ\cdot\nabla\mathbf{w}\cdot\mathbf{e}_k^{\circ\prime})\Big]\,d\tau$$

Here o_1 is the part of the surface on which the external forces are given (on $o_2\,\mathbf{w} = 0$).

The following equation of state [1] corresponds to the elastic potential (1.1):

$$\mathrm{D} = (\lambda s_1 - 2\mu)\,\mathrm{A} + 2\mu\nabla\mathbf{R} \tag{1.7}$$

where D is the Piola stress tensor. Therefore Π_1 takes the form

$$\Pi_1 = \iiint\limits_v \mathrm{D}^{\circ T}\cdot\cdot\nabla\mathbf{w}\,d\tau - \iint\limits_{o_1} \mathbf{F}^\circ\cdot\mathbf{w}\,do$$

Applying the easily verifiable identity

$$\mathrm{P}\cdot\cdot\nabla\mathbf{a} = \nabla\cdot(\mathrm{P}^T\cdot\mathbf{a}) - (\nabla\cdot\mathrm{P}^T)\cdot\mathbf{a} \tag{1.8}$$

and integrating by parts, we obtain

$$\Pi_1 = \iiint\limits_v -(\nabla\cdot\mathrm{D}^\circ)\cdot\mathbf{w}d\tau + \iint\limits_{o_1}(\mathbf{n}\cdot\mathrm{D}^\circ - \mathbf{F}^\circ)\cdot\mathbf{w}\,do \equiv 0$$

since the initial state of stress will be an equilibrium state and will satisfy the equations $\nabla\cdot\mathrm{D}^\circ = 0$ in the volume v and $\mathbf{n}\cdot\mathrm{D}^\circ = \mathbf{F}^\circ$ on the surface o_1.

Consequently, the growth increment of potential energy of deformation computed with accuracy to small terms of second order of smallness will be a homogeneous quadratic functional over the vector $\mathbf{w}$.

Further we shall demonstrate that the condition of this functional to be stationary, is equivalent to differential equations of neutral equilibrium with corresponding boundary conditions.

Let us construct the variation of the functional Π_2

$$\delta\Pi_2 = \iiint\limits_v [\lambda\nabla\delta\mathbf{w}\cdot\cdot\mathrm{A}^{\circ T}\nabla\mathbf{w}\cdot\cdot\mathrm{A}^{\circ T} + 2\mu\nabla\mathbf{w}\cdot\cdot\nabla\delta\mathbf{w} +$$
$$+\frac{1}{2}\frac{\lambda s_1^\circ - 2\mu}{\sqrt{G_s^\circ}+\sqrt{G_k^\circ}}\nabla\delta\mathbf{w}\cdot\cdot(\mathbf{e}_s^{\circ\prime}\mathbf{e}_k^\circ - \mathbf{e}_k^{\circ\prime}\mathbf{e}_s^\circ)\,\mathbf{e}_k^\circ\cdot\nabla\mathbf{w}\cdot\mathbf{e}_s^{\circ\prime} +$$
$$+\frac{1}{2}\frac{\lambda s_1^\circ - 2\mu}{\sqrt{G_s^\circ}+\sqrt{G_k^\circ}}\nabla\delta\mathbf{w}\cdot\cdot\mathbf{e}_s^{\circ\prime}\mathbf{e}_k^\circ\,(\mathbf{e}_k^\circ\cdot\nabla\mathbf{w}\cdot\mathbf{e}_s^{\circ\prime} - \mathbf{e}_s\cdot\nabla\mathbf{w}\cdot\mathbf{e}_k^{\circ\prime})\Big]\,d\tau$$

Integrating by parts, we further obtain, using (1.8)

$$\delta\Pi_2 = \iint\limits_o\Big\{\lambda\nabla\mathbf{w}^T\cdot\cdot\mathrm{A}^\circ\mathbf{n}\cdot\mathrm{A}^\circ\cdot\delta\mathbf{w} + 2\mu\mathbf{n}\cdot\nabla\mathbf{w}\cdot\delta\mathbf{w} +$$
$$+\mathbf{n}\left[\frac{\lambda s_1^\circ - 2\mu}{\sqrt{G_s^\circ}+\sqrt{G_k^\circ}}\nabla\mathbf{w}\cdot\cdot\mathbf{e}_s^{\circ\prime}\mathbf{e}_k^\circ\,(\mathbf{e}_k^\circ\mathbf{e}_s^{\circ\prime} - \mathbf{e}_s^\circ\mathbf{e}_k^{\circ\prime})\right]\cdot\delta\mathbf{w}\Big\}\,do -$$
$$-\iiint\limits_v\nabla\cdot\{\lambda\nabla\mathbf{w}^T\cdot\cdot\mathrm{A}^\circ\mathrm{A}^\circ + 2\mu\nabla\mathbf{w} + \frac{\lambda s_1^\circ - 2\mu}{\sqrt{G_s^\circ}+\sqrt{G_k^\circ}}\nabla\mathbf{w}\cdot\cdot\mathbf{e}_s^{\circ\prime}\mathbf{e}_k^\circ\,(\mathbf{e}_k^\circ\mathbf{e}_s^{\circ\prime} -$$
$$- \mathbf{e}_s^\circ\mathbf{e}_k^{\circ\prime})\}\cdot\delta\mathbf{w}d\tau$$

The requirement $\delta\Pi_2 = 0$ leads to differential equations of neutral equilibrium and boundary conditions [1]

$$\nabla\cdot\mathrm{D}^{\cdot} = 0 \ \text{ in the volume } v \quad \mathbf{n}\cdot\mathrm{D}^{\cdot} = 0 \ \text{ on the surface } o_1 \tag{1.9}$$

$$\mathrm{D}^{\cdot} = \frac{\lambda s_1^\circ - 2\mu}{\sqrt{G_s^\circ}+\sqrt{G_k^\circ}}\mathbf{e}_k^\circ\cdot\nabla\mathbf{w}\cdot\mathbf{e}_s^{\circ\prime}(\mathbf{e}_k^\circ\mathbf{e}_s^{\circ\prime} - \mathbf{e}_s^\circ\mathbf{e}_k^{\circ\prime}) + \lambda\mathrm{A}^\circ\mathbf{e}_k^\circ\cdot\nabla\mathbf{w}\cdot\mathbf{e}_k^{\circ\prime} + 2\mu\nabla\mathbf{w} \tag{1.10}$$

We note that $\mathbf{w} = \delta\mathbf{w} = 0$ on o_2. The converse statement is also apparent, when these conditions are satisfied $\delta\Pi_2 = 0$.

Now, keeping in mind Eq. (1.6), we can write functional Π_2 in the form

$$\Pi_2 = \frac{1}{2}\iiint_v D^{\cdot}\cdot\cdot\nabla \mathbf{w}^T d\tau \tag{1.11}$$

2. The case of affine initial deformation. In this case all quantities relating to the initial deformed state are constant and equations of the neutral equilibrium (1.9) and (1.10) can be simplified. Taking into account that

$$\mathbf{e}_s^{\circ\prime} = \mathbf{e}_s^{\circ}\cdot A^{\circ} = A^{\circ T}\cdot \mathbf{e}_s^{\circ}$$

we rewrite (1.10) as follows:

$$D^{\cdot\cdot} = \frac{\lambda s_1^{\circ} - 2\mu}{\sqrt{G_s^{\circ}} + \sqrt{G_k^{\circ}}}\mathbf{e}_k^{\circ}\cdot\nabla\mathbf{w}'\cdot\mathbf{e}_s^{\circ}\,(\mathbf{e}_k^{\circ}\mathbf{e}_s^{\circ} - \mathbf{e}_s^{\circ}\mathbf{e}_k^{\circ}) + \lambda E\mathbf{e}_k^{\circ}\cdot\nabla\mathbf{w}'\cdot\mathbf{e}_k^{\circ} + 2\mu\nabla\mathbf{w}' \tag{2.1}$$

$$D^{\cdot\cdot} = (D^{\cdot})\cdot A^{\circ T},$$

Here $\mathbf{w}' = \mathbf{w}\cdot A^{\circ T}$ is the rotated vector of displacement. It is apparent that Eqs. (1.9) are equivalent to equations

$$\nabla\cdot D^{\cdot\cdot} = 0 \text{ in } v,\ \mathbf{n}\cdot D^{\cdot\cdot} = 0 \text{ on } o_1 \tag{2.2}$$

Through transformations analogous to those carried out in [1] we can reduce (2.1) to the form

$$D^{\cdot\cdot} = T(\mathbf{w}') - 2\mu E \times (C\cdot\boldsymbol{\omega}') \tag{2.3}$$

where

$$T(\mathbf{w}') = \lambda E\nabla\cdot\mathbf{w}' + 2\mu\boldsymbol{\varepsilon}',\quad \boldsymbol{\varepsilon}' = {}^1\!/_2\,(\nabla\mathbf{w}' + \nabla\mathbf{w}'^T),\quad \boldsymbol{\omega}' = {}^1\!/_2\nabla\times\mathbf{w}' \tag{2.4}$$

$$C = E + \frac{1}{\mu}\,\frac{\lambda s_1^{\circ} - 2\mu}{\sqrt{G_3^{\circ}} + \sqrt{G_2^{\circ}}}\,\mathbf{e}_1^{\circ}\mathbf{e}_1^{\circ} + \frac{1}{\mu}\,\frac{\lambda s_1^{\circ} - 2\mu}{\sqrt{G_3^{\circ}} + \sqrt{G_1^{\circ}}}\,\mathbf{e}_2^{\circ}\mathbf{e}_2^{\circ} + $$
$$+ \frac{1}{\mu}\,\frac{\lambda s_1^{\circ} - 2\mu}{\sqrt{G_1^{\circ}} + \sqrt{G_2^{\circ}}} \times \mathbf{e}_3^{\circ}\mathbf{e}_3^{\circ}$$

In this manner it is shown that Eqs. (2.2) and (2.3) obtained in [1] for the case of triaxial uniform elongation are applicable to any affine transformation, only instead of the actual vector of displacement it is necessary to take as $\mathbf{w}$ the rotated vector of displacement.

The functional (1.1) is also represented through the rotated vector $\mathbf{w}'$

$$\Pi_2 = \frac{1}{2}\iiint_v (D^{\cdot\cdot})\cdot\cdot\nabla\mathbf{w}'^T d\tau$$

To avoid cumbersome notation in the following we shall omit the prime above vector $\mathbf{w}$ and tensor $D^{\cdot}$. It is agreed that $\mathbf{w}$ and $D^{\cdot}$ will be understood to be $\mathbf{w}\cdot A^{\circ T}$ and $(D^{\cdot})\cdot A^{\circ T}$, respectively.

The growth increment of potential energy Π_2 can be expressed through components of tensor $D^{\cdot}$. For this purpose the tensor $D^{\cdot}$ will be represented in a form composed of a symmetrical and an antisymmetrical part

$$D^{\cdot} = T - E\times\mathbf{q}$$

where $\mathbf{q}$ is the vector which goes along with tensor $D^{\cdot}$. Comparing with (2.3) we obtain

$$2\mu\nabla\mathbf{w} = T - \frac{\nu}{1+\nu}\sigma E - E\times(C^{-1}\cdot\mathbf{q}),\quad \sigma = I_1(T),\quad \nu = \frac{\lambda}{2(\lambda+\mu)}$$

Then using the equality

$$I_1(\mathbf{a}\times E\times\mathbf{b}) = -2\mathbf{a}\cdot\mathbf{b}$$

we arrive at the desired expression

$$\Pi_2 = \frac{1}{2}\iiint_v (D^{\cdot})\cdot\cdot\nabla\mathbf{w}^T d\tau = \iiint_v \frac{1}{4\mu}\left[T\cdot\cdot T - \frac{\nu}{1+\nu}\sigma^2 + 2\mathbf{q}\cdot C^{-1}\cdot\mathbf{q}\right] d\tau$$

To the variational principle established in Sect. 1, we can give the formulation of a mixed principle analogous to the principle of Reissner in the classical theory of elasticity. Namely, let us examine the expression

$$\Phi = \iiint_v [(\mathrm{D}^\cdot)\cdot\cdot\nabla \mathbf{w}^T - \frac{1}{4\mu}(\mathrm{T}\cdot\cdot\mathrm{T} - \frac{\nu}{1+\nu}\sigma^2 + 2\mathbf{q}\cdot\mathrm{C}^{-1}\cdot\mathbf{q})]\,d\tau \tag{2.5}$$

where Φ is the functional over vector $\mathbf{w}$ and tensor $\mathrm{D}^\cdot$, which are examined as independent functions of the coordinates. The numerical value of Φ is equal to Π_2. It is easy to check that the requirement of stationarity of this functional leads to equations of neutral equilibrium and boundary conditions on o_1 written in components of tensor $\mathrm{D}^\cdot$ (the comparison includes vectors $\mathbf{w}$ which satisfy the condition $\mathbf{w} = 0$ on o_2) and also equations which connect tensor $\nabla\mathbf{w}$ with tensor $\mathrm{D}^\cdot$

$$\nabla\cdot\mathrm{D} = 0 \quad \text{in } v, \qquad \mathbf{n}\cdot\mathrm{D}^\cdot = 0 \quad \text{on } o_1$$

$$2\mu\varepsilon = \mathrm{T} - \frac{\nu}{1+\nu}\sigma\mathrm{E}, \qquad 2\mu\omega = \mathrm{C}^{-1}\cdot\mathbf{q}$$

3. Derivation of two-dimensional equations of buckling of plates. Let us presume that a thin plate is subjected to affine transformation in its own plane, accompanied by uniform elongation along the z-axis. Such an initial deformation is realized for example in a rectangular plate when its side surfaces are loaded by forces located in the plane of the plate and which have a constant intensity along each pair of opposing sides. For the described deformation the z-axis will be the principal axis of deformation, i.e. $\mathbf{e}_3^\circ = \mathbf{e}_3^{\circ\prime} = \mathbf{i}_3$. In this case the forms of bifurcation of plate equilibrium according to (2.2) and (2.3) fall into two independent parts: symmetrical with respect to the mean plane $z = 0$, and antisymmetrical. If components of displacement along the axes x, y and z are designated by u_1, u_2 and w, then for the antisymmetrical, i.e. bending forms, the component w will be an even function of z, and u_1 and u_2 will be uneven functions of z.

Components $\partial_{13}^\cdot$, $\partial_{23}^\cdot$, $\partial_{31}^\cdot$ and $\partial_{32}^\cdot$ of tensor $\mathrm{D}^\cdot$ will be even functions of z. The remaining components will be uneven functions of z.

By a method which is analogous to the derivation of equations in the theory of shells from the principle of Reissner in [3], we obtain approximate two-dimensional equations which describe the bending forms of bifurcation of the plate from the mixed variational principle formulated in Sect. 2.

For the bending forms of bifurcation of a thin plate let us write the following approximation of the vector of displacement $\mathbf{w}$ and the tensor $\mathrm{D}^\cdot$ as a function of coordinate z:

$$\mathbf{w} = \mathbf{w}_1 z + w_0\mathbf{i}_3, \qquad \mathbf{w}_1 = u_1\mathbf{i}_1 + u_2\mathbf{i}_2 \tag{3.1}$$

$$\mathrm{D}^\cdot = \frac{12}{h^3}\mathrm{M}z + \frac{3}{4h}\left[1 - \left(\frac{2z}{h}\right)^2\right]\mathbf{V}_1\mathbf{i}_3 + \frac{1}{2h}\mathbf{V}_3\mathbf{i}_3 + \frac{3}{2h}\left[1 - \left(\frac{2z}{h}\right)^2\right]\mathbf{i}_3\mathbf{V}_2$$

Here M is a two-dimensional tensor $\mathbf{w}_1$, $\mathbf{V}_1$, $\mathbf{V}_2$ and $\mathbf{V}_3$ are two-dimensional vectors. The component $\partial_{33}^\cdot$ is neglected because for $z = \pm\,{}^1/_2 h$ this component is equal to 0, and furthermore $\partial_{33}^\cdot$ is an uneven function of z. In the approximation adopted in (3.1) the condition $\mathbf{n}\cdot\mathrm{D}^\cdot = 0$ for $z = \pm\,{}^1/_2 h$ is satisfied. The integral meaning of introduced quantities is determined by the following equations:

$$\mathrm{M} = \int_{-{}^1/_2 h}^{{}^1/_2 h} \partial_{sk}^\cdot \mathbf{i}_s\mathbf{i}_k z\,dz, \quad {}^1/_2(\mathbf{V}_1 + \mathbf{V}_3) = \int_{-{}^1/_2 h}^{{}^1/_2 h} \partial_{s3}^\cdot \mathbf{i}_s\,dz, \quad \mathbf{V}_2 = \int_{-{}^1/_2 h}^{{}^1/_2 h} \partial_{3s}^\cdot \mathbf{i}_s\,dz \qquad (s, k = 1,2)$$

Substituting expressions (3, 1) into (2, 5), we obtain after transformations and integration with respect to z

$$\Phi = \iint_S \{\mathbf{M}_c \cdot\cdot \varepsilon_1 + \frac{1}{2}\left(\frac{1}{2}\mathbf{V}_1 + \mathbf{V}_2\right)\cdot(\nabla w_0 + \mathbf{w}_1) + \frac{1}{4}\mathbf{V}_3\cdot(\nabla w_0 + \mathbf{w}_1) + 2\omega_1\cdot$$

$$\mathbf{q}_M + {}^1/_2({}^1/_2\mathbf{V}_1 - \mathbf{V}_2)\cdot(\nabla w_0 - \mathbf{w}_1) + {}^1/_4\mathbf{V}_3\cdot(\nabla w_0 - \mathbf{w}_1) - \frac{1}{4\mu}\left[\frac{12}{h^3}\mathbf{M}_c\cdot\cdot\mathbf{M}_c + \right.$$

$$+ \frac{3}{5h}\left(\frac{1}{2}\mathbf{V}_1 + \mathbf{V}_2\right)\cdot\left(\frac{1}{2}\mathbf{V}_1 + \mathbf{V}_2\right) + \frac{1}{8h}\mathbf{V}_3\cdot\mathbf{V}_3 + \frac{1}{2h}({}^1/_2\mathbf{V}_1 + \mathbf{V}_2)\cdot\mathbf{V}_3 -$$

$$\frac{\nu}{1+\nu}\frac{12}{h^3}\sigma_M{}^2 + \frac{24}{h^3}\mathbf{q}_M\cdot\mathbf{C}^{-1}\cdot\mathbf{q}_M + \frac{3}{5h}\left(\frac{1}{2}\mathbf{V}_1 - \mathbf{V}_2\right)\cdot\mathbf{C}_1^{-1}\cdot\left(\frac{1}{2}\mathbf{V}_1 - \mathbf{V}_2\right) +$$

$$\left. + \frac{1}{8h}\mathbf{V}_3\cdot\mathbf{C}_1^{-1}\cdot\mathbf{V}_3 + \frac{1}{2h}\left(\frac{1}{2}\mathbf{V}_1 - \mathbf{V}_2\right)\cdot\mathbf{C}_1^{-1}\cdot\mathbf{V}_3\right]\} do$$

$$\mathbf{M} = \mathbf{M}_c - \mathbf{E}_2\times\mathbf{q}_M$$

$$\sigma_M = I_1(\mathbf{M}), \quad \varepsilon_1 = {}^1/_2(\nabla\mathbf{w}_1 + \nabla\mathbf{w}_1{}^T), \quad \omega_1 = {}^1/_2\nabla\times\mathbf{w}_1$$

$$\mathbf{C}_1 = C_2\mathbf{e}_1{}^\circ\mathbf{e}_1{}^\circ + C_1\mathbf{e}_2{}^\circ\mathbf{e}_2{}^\circ = (\mathbf{e}_1{}^\circ\mathbf{e}_2{}^\circ + \mathbf{e}_2{}^\circ\mathbf{e}_1{}^\circ)\cdot\mathbf{C}\cdot(\mathbf{e}_1{}^\circ\mathbf{e}_2{}^\circ + \mathbf{e}_2{}^\circ\mathbf{e}_1{}^\circ)$$

Here S is the mean plane of the plate; the symbol ∇ is undestood to represent now the two-dimensional nabla operator; $\mathbf{M}_c$ is the symmetrical part of the tensor $\mathbf{M}$; C_1, C_2 and C_3 are the principal components of tensor $\mathbf{C}$.

Computation of variation $\delta\Phi$ and integration by parts gives

$$\delta\Phi = \iint_S \{\delta w_0[-{}^1/_2\nabla\cdot\mathbf{V}_1 - {}^1/_2\nabla\cdot\mathbf{V}_3] + \delta\mathbf{w}_1\cdot[-\nabla\cdot\mathbf{M}_c + \nabla\times\mathbf{q}_M + \mathbf{V}_2] +$$

$$+ \delta\mathbf{M}_c\cdot\cdot\left[\varepsilon_1 - \frac{1}{2\mu}\left(\frac{12}{h^3}\mathbf{M}_c - \frac{\nu}{1+\nu}\frac{12}{h^3}\sigma_M\mathbf{E}_2\right)\right] + \delta\mathbf{q}_M\cdot\left[2\omega_1 - \frac{12}{h^3\mu}\mathbf{C}^{-1}\cdot\mathbf{q}_M\right] +$$

$$+ \delta\mathbf{V}_1\cdot[{}^1/_2\nabla w_0 - \frac{1}{4\mu}\left\{\frac{3}{5h}({}^1/_2\mathbf{V}_1 + \mathbf{V}_2) + \frac{1}{4h}\mathbf{V}_3 + \frac{3}{5h}\mathbf{C}_1^{-1}\cdot({}^1/_2\mathbf{V}_1 - \mathbf{V}_2) + \right.$$

$$\left. + \frac{1}{4h}\mathbf{C}_1^{-1}\cdot\mathbf{V}_3\right\}] + \delta\mathbf{V}_2\cdot\left[\mathbf{w}_1 - \frac{1}{4\mu}\left\{\frac{6}{5h}({}^1/_2\mathbf{V}_1 + \mathbf{V}_2) + \frac{1}{2h}\mathbf{V}_3 + \frac{6}{5h}\mathbf{C}_1^{-1}\cdot\left(\mathbf{V}_2 - \right.\right.\right.$$

$$\left.\left.\left. - \frac{1}{2}\mathbf{V}_1\right) - \frac{1}{2h}\mathbf{C}_1^{-1}\cdot\mathbf{V}_3\right\}\right] + \delta\mathbf{V}_3\cdot[{}^1/_2\nabla w_0 - \frac{1}{4\mu}\left\{\frac{1}{4h}\mathbf{V}_3 + \frac{1}{2h}\left(\frac{1}{2}\mathbf{V}_1 + \mathbf{V}_2\right) + \right.$$

$$\left. + \frac{1}{4h}\mathbf{C}_1^{-1}\cdot\mathbf{V}_3 + \frac{1}{2h}\left(\frac{1}{2}\mathbf{V}_1 - \mathbf{V}_2\right)\cdot\mathbf{C}_1^{-1}\right\}]\} do + \oint_\gamma [\mathbf{n}\cdot\mathbf{M}_c\cdot\delta\mathbf{w}_1 - (\mathbf{n}\times\mathbf{q}_M)\cdot\delta\mathbf{w}_1 +$$

$$+ {}^1/_2\mathbf{n}\cdot(\mathbf{V}_1 + \mathbf{V}_3)\,\delta w_0]\, ds \tag{3.2}$$

Here γ is the contour which forms the boundary of the mean plane of the plate; $\mathbf{n}$ is the normal to it. From here, using the condition of independence of variation, we arrive at the equations of equilibrium

$$\nabla\cdot\mathbf{M} = \mathbf{V}_2, \qquad \nabla\cdot(\mathbf{V}_1 + \mathbf{V}_3) = 0 \tag{3.3}$$

and relationships which connect $\mathbf{M}$, $\mathbf{V}_1$, $\mathbf{V}_2$ and $\mathbf{V}_3$ with kinematic quantities

$$\mathbf{M} = \frac{2\mu h^3}{12}\left[\varepsilon_1 + \frac{\nu}{1-\nu}\nabla\cdot\mathbf{w}_1\mathbf{E}_2 - \mathbf{E}_2\times(\mathbf{C}\cdot\omega_1)\right]$$

$$\mathbf{V}_1 = {}^5/_3\mu h(\mathbf{E}_2 - \mathbf{C}_1)\cdot[\mathbf{w}_1 + \nabla w_0\cdot(\mathbf{E}_2 - \mathbf{C}_1)\cdot(\mathbf{E}_2 + \mathbf{C}_1)^{-1}] \tag{3.4}$$

$$\mathbf{V}_2 = {}^5/_6\mu h[\nabla w_0\cdot(\mathbf{E}_2 - \mathbf{C}_1) + \mathbf{w}_1\cdot(\mathbf{E}_2 + \mathbf{C}_1)]$$

$$\mathbf{V}_3 = 8\mu h\nabla w_0\cdot\mathbf{C}_1\cdot(\mathbf{E}_2 + \mathbf{C}_1)^{-1}$$

Different variants of boundary conditions are apparent from the structure of the contour integral in (3.2). In the absence of an initial state of stress, i.e. for $C = 0$, relationships (3.3) and (3.4) transform into known equations of Reissner's theory of plates in the absence of transverse loading on the plate [4]. Substituting (3.4) into (3.3) we obtain equations in displacements describing the buckling of the plate

$$\nabla\cdot(\mathrm{E}_2 - \mathrm{C}_1)\cdot\mathbf{w}_1 + \nabla\cdot(\mathrm{E}_2 + \mathrm{C}_1)^{-1}\cdot(\mathrm{E}_2 + {}^{14}/_5\mathrm{C}_1 + \mathrm{C}_1{}^2)\cdot\nabla w_0 = 0 \qquad (3.5)$$

$$^1/_5 h^2\left[\left(\frac{\nu}{1-\nu} + \frac{1-\mathrm{C}_3}{2}\right)\nabla\nabla\cdot\mathbf{w}_1 + \frac{1+\mathrm{C}_3}{2}\nabla^2\mathbf{w}_1\right] = \nabla w_0\cdot(\mathrm{E}_2 - \mathrm{C}_1) + \mathbf{w}_1\cdot$$
$$\cdot(\mathrm{E}_2 + \mathrm{C}_1)$$

4. Example. As a most simple example we shall examine the axisymmetric forms of buckling of a circular plate compressed along the contour by a uniformly distributed normal pressure. The edge of the plate can freely move along the z-axis but is fixed against rotation.

In the undeformed plate let us introduce cylindrical coordinates r, θ and z and the corresponding basis vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{i}_3$. The radius vector of a point after deformation is given in the form

$$\mathbf{R}^\circ = \beta r\mathbf{e}_r + \alpha z\mathbf{i}_3$$

From (1.7) we determine the Piola stress tensor corresponding to this deformation

$$\mathrm{D}^\circ = [\lambda\,(2\beta + \alpha - 3) - 2\mu]\,\mathrm{E} + 2\mu\,[\beta\mathrm{E}_2 + \alpha\mathbf{i}_3\mathbf{i}_3]$$

We select α and β from conditions $\partial_{rr}{}^\circ = -p_1$ and $\partial_{33}{}^\circ = 0$

$$\beta = 1 - \frac{1-\nu}{1+\nu}p_1{}^*, \quad \alpha = 1 + \frac{2\nu}{1+\nu}p_1{}^*, \quad p_1{}^* = \frac{p_1}{2\mu}, \quad 0 \leqslant p_1{}^* < \frac{1+\nu}{1-\nu}$$

Here p_1 is the pressure computed per unit area of the undeformed body. The true pressure, i.e. the pressure computed per unit area of the deformed body, is equal to

$$\frac{p}{2\mu} = p^* = \frac{p_1{}^*}{\alpha\beta} = p_1{}^*\left[\left(1 - \frac{1-\nu}{1+\nu}p_1{}^*\right)\left(1 + \frac{2\nu}{1+\nu}p_1{}^*\right)\right]^{-1} \qquad (4.1)$$

It is easy to see that p is a monotone function of p_1 so that minimum p corresponds to minimum p_1; according to (2.4) we shall have

$$\mathrm{C} = \frac{-p_1{}^*}{2 - p_1{}^*(1-3\nu)/(1+\nu)}\mathrm{E}_2 - \frac{p_1{}^*}{1 - p_1{}^*(1-\nu)/(1+\nu)}\mathbf{i}_3\mathbf{i}_3 = \mathrm{C}_1\mathrm{E}_2 + \mathrm{C}_3\mathbf{i}_3\mathbf{i}_3$$

In the example under examination the system (3.5) assumes the form

$$\begin{gathered}(1-\mathrm{C}_1)\left(u' + \frac{u}{r}\right) + \frac{1 + {}^{14}/_5\mathrm{C}_1 + \mathrm{C}_1{}^2}{1+\mathrm{C}_1}\left(w_0'' + \frac{w_0'}{r}\right) = 0 \\ \frac{h^2}{5(1-\nu)}\left(u' + \frac{u}{r}\right)' = (1-\mathrm{C}_1)\,w_0' + (1+\mathrm{C}_1)\,u \quad (u = \mathbf{w}\cdot\mathbf{e}_r)\end{gathered} \qquad (4.2)$$

The system (4.2) has the following solution:

$$u = \frac{A}{k}J_1(kr), \quad w_0 = -\frac{A(1-\mathrm{C}_1)h^2}{24\mathrm{C}_1(1-\nu)}J_0(kr) + B$$

where

$$h^2k^2 = \frac{-24\mathrm{C}_1(1+\mathrm{C}_1)(1-\nu)}{1 + {}^{14}/_5\mathrm{C}_1 + \mathrm{C}_1{}^2} =$$
$$= 48(1-\nu)\,p_1{}^*\left(1 - \frac{1-\nu}{1+\nu}p_1{}^*\right)\left[4 - \frac{16}{5}\frac{3-2\nu}{1+\nu}p_1{}^* + \frac{8}{5}\frac{3-6\nu+\nu^2}{(1+\nu)^2}p_1{}^{*2}\right]^{-1}$$

From the boundary condition $u = 0$ for $r = a$ for the nontrivial solution we arrive at the transcendental equation $J_1(ka) = 0$. For the critical values of external pressure the following equation is obtained

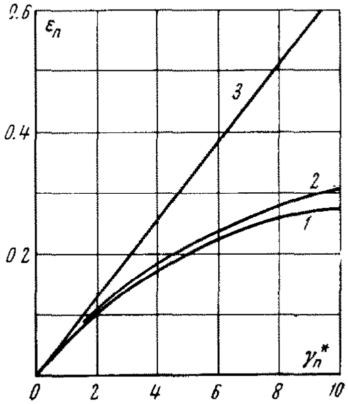

Fig. 1

$$\left[\frac{8}{5}\,\frac{3-6\nu+\nu^2}{(1+\nu)^2}\,\frac{h^2}{a^2}\,\gamma_n^2+\frac{48\,(1-\nu)^2}{1+\nu}\right]p_1^{*2}-$$
$$-\left[\frac{16}{5}\,\frac{3-2\nu}{1+\nu}\,\frac{h^2}{a^2}\,\gamma_n^2+48\,(1-\nu)\right]p_1^*+$$
$$+4\,\frac{h^2}{a^2}\,\gamma_n^2=0$$

where γ_n are zeros of the Bessel function $J_1(\gamma_n)=0$.

In this case the second boundary condition

$$\mathbf{e}_r\cdot(\mathbf{V}_1+\mathbf{V}_3)=0 \quad \text{for } r=a$$

is also satisfied.

In Fig. 1 the curve 1 represents the relationship $\varepsilon_n=\varepsilon_n(\gamma_n^*)$ for $\nu=0.3$ of the critical relative shortening of the plate radius

$$\varepsilon_n=1-\beta_n=\frac{1-\nu}{1+\nu}\,p_{1n}^*,\quad \gamma_n^*=\gamma_n^2\,\frac{h^2}{a^2}$$

The curve 2 corresponds to the exact solution of axisymmetric bifurcation of equilibrium of a circular cylinder compressed on the lateral surface by a uniform pressure. This result was obtained in [5].

The straight line 3 corresponds to the classical linear theory of buckling plates.

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ON THE CORRECTNESS OF CERTAIN PROBLEMS OF THE MEMBRANE THEORY OF SHELLS OF NEGATIVE CURVATURE

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V. A. SHACHNEV
(Moscow)
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The determination of the state of stress and strain of a membrane shell of negative curvature reduces to the requirement of solving a system of hyperbolic-type equations. The boundary value problem for such a system does not always have a solution, and hence, such a problem is not generally correct. The following boundary value problem will be examined herein for the system of membrane theory equations in the case of shells of